A NOTE ON TWO WEIGHT BOUNDS FOR THE GENERALIZED HARDY-LITTLEWOOD MAXIMAL OPERATOR

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ABSTRACT. We give a straighforward proof of the two weight estimates of the generalized maximal operator under Sawyer type testing conditions. The proof relies on the Martingale Carleson Embedding Theorem.

1. INTRODUCTION

Let (X, Σ, ω) be a σ-finite measure space with a filtration $\Sigma_n, n \in \mathbb{Z}$ (an increasing sequence of σ-algebras with $\Sigma_n \subset \Sigma$). We make the assumption that for each σ-algebra Σ_n there exists a countable collection $\mathcal{D}_n$ of disjoint sets of finite measure such that any set in Σ_n can be written as a union of sets in $\mathcal{D}_n$. The elements of any $\mathcal{D}_n$ are pairs of the form (Q, n). However, in what follows, by slightly abusing notation, we will ignore the dependence on n and simply write Q instead.

Let $\mathcal{D} = \bigcup_n \mathcal{D}_n$. A classical example of such a filtration is that given by a dyadic lattice on $\mathbb{R}^n$ and in fact we may often refer to $\mathcal{D}$ as a lattice and to its elements as cubes. In spite of this language, we will not be making any further assumptions on the underlying structure of the space, including for example any assumptions about the homogeneity of X with the measure ω.

Let μ and ν be measures, finite on all $Q \in \mathcal{D}$.

For a sequence of functions $a = \{a_Q\}_{Q \in \mathcal{D}}$, $a_Q : X \to [0, \infty)$ indexed by the sequence of dyadic cubes, define the operator M_a^q, given by

$$M_a^q f\mu(x) = \left(\sum_{\substack{Q \in \mathcal{D} \\ x \in Q}} \left| \left(\int_Q f d\mu \right) a_Q(x) \mathbb{1}_Q(x) \right|^q \right)^{1/q} \quad \text{for } 1 < q < \infty$$

and

$$M_a^\infty f\mu(x) = \sup \left\{ \left| \left(\int_Q f d\mu \right) a_Q(x) \mathbb{1}_Q(x) \right| : Q \in \mathcal{D} \quad \text{such that } x \in Q \right\}.$$

In what follows we will show that under a so-called Sawyer type testing condition, the operator $f \mapsto M_a^q f\mu$ is bounded $L^p(\mu) \to L^p(\nu)$ for $p \leq q$. Such conditions were named after E. Sawyer, who introduced them in [6] when studying the two weight estimates for the classical maximal operator M. The testing condition presented in [6] essentially amounts to testing the uniform estimates on characteristic functions of dyadic cubes. Later, in [7], Sawyer proved that for operators such as fractional integrals, Poisson kernels, and other nonnegative kernels, the two weight estimate still holds if one assumes the testing condition

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not only on the operator itself, but also on its formal adjoint. For the positive martingale operators such results were obtained in [4] ($p = 2$) and later in [1] (general p), see also [8] for a simpler proof.

The main result of this paper is a simple proof of the theorem below. Formally, this result is new, because we allow a_Q to be nonnegative functions and we also consider the case $q < \infty$.

For $Q \in \mathcal{D}$ define the truncated operator $M^q_{a,Q}$,

$$M^q_{a,Q} f\mu(x) := \left(\sum_{\substack{R \in \mathcal{D}, R \subset Q \\ x \in R}} \left| \left(\int_R f d\mu \right) a_R(x) \mathbb{1}_R(x) \right|^q \right)^{1/q},$$

with the obvious modification for $q = \infty$.

Theorem 1. *Let $1 < p \leq q \leq \infty$. The operator M^q_a satisfies*

(1.1) $$\|M^q_a f\mu\|^p_{L^p(\nu)} \leq A \|f\|_{L^p(\mu)} \qquad \forall f \in L^p(\mu)$$

if and only if the truncation $M^q_{a,Q}$ satisfies the following testing condition

(1.2) $$\|M^q_{a,Q}(\mathbb{1}_Q \mu)\|_{L^p(\nu)} \leq B\mu(Q)^{1/p}, \text{ for any } Q \in \mathcal{D}.$$

Moreover, for the best constants A and B we have $B \leq A \leq C(p)B$,

$$C(p) = \left((1 + 1/p)^{p+1} p \right)^{1/p} p',$$

where p' is the Hölder conjugate of p, $1/p + 1/p' = 1$.

For $p = q = \infty$, the result is trivial with $A = B$. Note that $\lim_{p \to \infty} C(p) = 1$.

Remark 1. The classical dyadic (martingale) maximal operator M is a particular case of operator M^q_a where $q = \infty$, $a_Q \equiv \omega(I)^{-1} \mathbb{1}_Q$, and $\mathcal{D}$ is a dyadic lattice in $\mathbb{R}^n$. Therefore, one can view the M^q_a as a generalization of the classical martingale maximal function.

In [6], E. Sawyer considered slightly more general maximal operators $M = M^\alpha$ which are a particular case of our M^q_a with $q = \infty$ and $a_Q \equiv \omega(I)^{-\alpha} \mathbb{1}_Q$, $0 < \alpha \leq 1$. He characterized the measures $\widetilde{\mu}$, ν and ω for which the inequality

(1.3) $$\|Mf\omega\|_{L^p(\nu)} \leq A \|f\|_{L^p(\widetilde{\mu})} \qquad \forall f \in L^p(\widetilde{\mu})$$

holds.

Note that without loss of generality one can assume that $\widetilde{\mu}$ is absolutely continuous with respect to ω, $d\widetilde{\mu} = w d\omega$ (adding a singular part to $\widetilde{\mu}$ does not change (1.3)). So, making the standard change of weight $f \mapsto w^{p'/p} f$ and denoting $\mu := w^{-p'/p} \omega$ we transform the above estimate (1.3) to (1.1). Then in this notation the necessary and sufficient condition obtained by E. Sawyer is exactly the testing condition (1.2).

For the classical dyadic maximal operator $M = M^\alpha$, the truncation $M^\infty_{a,Q} \mathbb{1}_Q \mu$ defined above is equivalent to $\mathbb{1}_Q M \mathbb{1}_Q \mu$, so up to a change of measure, our setup is identical to [6].

Remark 2. The reduction to the two measure setup, eliminating the underlying measure ω, is now considered standard for weighted estimates. In the three measures setup for the classical maximal operator as in [6] all the information about ω is captured by the (constant in this case) functions a_Q.

To obtain Sawyer's estimate for the non-martingale maximal function, one can use the two weighted estimate for the dyadic case and proceed by an averaging argument. This reasoning is pretty standard and we will not discuss it at all in this paper.

Our proof simplifies the argument in [6] and gives a formally stronger result: in particular, the coefficients a_Q do not need to be constants. It also has the additional benefit of placing the Hardy-Littlewood maximal function in the context of a wide range of similar operators.

The proof we present relies on the stopping time construction presented in [8] and the Martingale Carleson Embedding Theorem stated below.

Denote
$$\fint_Q f d\mu = \frac{1}{\mu(Q)} \int_Q f d\mu.$$

Theorem 2. *(Martingale Carleson Embedding Theorem) Let μ be a measure on X and let $\{w_Q\}_{Q \in \mathcal{D}}$, $w_Q \geq 0$ be a sequence satisfying the following condition:*
$$\sum_{Q \subset R, R \in \mathcal{D}} w_Q \leq A\mu(R), \text{ for any cube } R \in \mathcal{D}.$$

Then for any measurable function $f \geq 0$ and for any $p \in (1, \infty)$,
$$\sum_{Q \in \mathcal{D}} \left(\fint_Q f d\mu \right)^p w_Q \leq (p')^p A \|f\|_{L^p(\mu)}^p.$$

The Carleson Embedding Theorem with the constant $(p')^p$ can be proved as a straightforward consequence of the one weight L^p boundedness of the classical Hardy-Littlewood maximal function (see [8]). Other arguments that include the sharp constant have been given by Nazarov, Treil, and Volberg [5] for $p = 2$ and Lai [2] for $p \neq 2$ using Bellman function techinques. The exact Bellman function for $p > 1$ was originally computed by Melas in [3], but the sharp constant was not explicitly stated.

2. Proof of the main result

We aim to prove Theorem 1. Again, we make the remark that while we may refer to the elements $Q \in \mathcal{D}$ as cubes, $\mathcal{D}$ is not necessarily assumed to be the dyadic lattice on $\mathbb{R}^n$.

Proof. The necessity of the testing condition and the estimate $B \leq A$ are trivial: if M_a^q is bounded on $L^p(\mu)$ functions, it is, in particular, bounded on caracteristic functions. Thus, by testing M_a^q on the functions $\mathbb{1}_Q$, we obtain condition (1.2).

To prove sufficiency, we begin by constructing a collection of stopping cubes $\mathcal{G} \subset \mathcal{D}$, following the definitions and notation in [8].

For any cube $Q \in \mathcal{D}$, define $\mathcal{D}(Q)$ to be the collection of subcubes of Q in $\mathcal{D}$. For a fixed $r > 1$, let $\mathcal{G}^*(Q)$ be the set of stopping cubes of Q, that is,
$$\mathcal{G}^*(Q) = \left\{ R \in \mathcal{D}(Q), R \text{ maximal}, \frac{1}{\mu(R)} \int_R f d\mu \geq r \frac{1}{\mu(Q)} \int_Q f d\mu \right\},$$

where maximality is considered with respect to the partial ordering given by inclusion.

Denote by $\mathcal{E}(Q)$ the collection of descendants of Q that are not stopping cubes or descendants of the stopping cubes:

$$\mathcal{E}(Q) = \mathcal{D}(Q) \setminus \bigcup_{P \in \mathcal{G}^*(Q)} \mathcal{D}(P).$$

Note that, by definition, for any $R \in \mathcal{E}(Q)$,

(2.1)
$$\fint_R f \, d\mu < r \fint_Q f \, d\mu.$$

Also note that

(2.2)
$$\sum_{R \in \mathcal{G}^*(Q)} \mu(R) = \mu \left(\bigcup_{R \in \mathcal{G}^*(Q)} R \right) \leq \frac{\mu(Q)}{r}.$$

To construct the collection $\mathcal{G}$ of stopping cubes, let N be a fixed large positive integer and define the first generation $\mathcal{G}_1$ as

$$\mathcal{G}_1 = \mathcal{D}_{-N}.$$

Then, to obtain the subsequent generations, apply the inductive formula

$$\mathcal{G}_{n+1} = \bigcup_{Q \in \mathcal{G}_n} \mathcal{G}^*(Q).$$

Define the collection of stopping cubes $\mathcal{G}$ to be the union

$$\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n.$$

Equation (2.2) implies that

(2.3)
$$\sum_{R \in \mathcal{G}, R \subset Q} \mu(R) \leq \frac{r}{r-1} \mu(Q), \ \forall Q \in \mathcal{D}.$$

To prove the theorem it is sufficient to prove the uniform in N bounds for the operator $M_a^{q,N}$,

$$M_a^{q,N} f := \left(\sum_{n \geq -N} \sum_{Q \in \mathcal{D}_n} \left(\left(\fint_R f \, d\mu \right) \mu(R) a_R(x) \mathbb{1}_R(x) \right)^q \right)^{1/q}$$

(with the obvious change for $q = \infty$) and then let $N \to \infty$.

Given the construction of stopping moments, it is easy to see that

$$\bigcup_{n=-N}^{\infty} \mathcal{D}_n = \bigcup_{Q \in \mathcal{G}} \mathcal{E}(Q).$$

and that the sets $\mathcal{E}(Q)$ are disjoint.

In the proof below we use notation for $1 < q < \infty$. The proof for $q = \infty$ is absolutely the same (up to obvious changes in the notation).

Denoting

$$F_Q(x) = \left(\sum_{R \in \mathcal{E}(Q)} \left(\left(\fint_R f \, d\mu \right) \mu(R) a_R(x) \mathbb{1}_R(x) \right)^q \right)^{1/q},$$

we can write $M_a^{q,N} f = \left(\sum_{Q\in\mathcal{G}} F_Q^q \right)^{1/q}$, so the proof amounts to bounding $\left\| \left(\sum_{Q\in\mathcal{G}} F_Q^q \right)^{1/q} \right\|_{L^p(\nu)}$.

Since $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ for $q \geq p$, we can estimate

(2.4)
$$\left\| \left(\sum_{Q\in\mathcal{G}} F_Q^q \right)^{1/q} \right\|_{L^p(\nu)} \leq \left\| \left(\sum_{Q\in\mathcal{G}} F_Q^p \right)^{1/p} \right\|_{L^p(\nu)}$$

$$= \left(\int \sum_{Q\in\mathcal{G}} F_Q^p \, d\nu \right)^{1/p}$$

$$= \left(\sum_{Q\in\mathcal{G}} \int F_Q^p \, d\nu \right)^{1/p}$$

$$= \left(\sum_{Q\in\mathcal{G}} \|F_Q\|_{L^p(\nu)}^p \right)^{1/p}.$$

By definition,

(2.5) $\|F_Q\|_{L^p(\nu)}^p = \left\| \left(\sum_{R\in\mathcal{E}(Q)} \left(\left(\fint_R f\,d\mu \right) \mu(R) a_R \mathbb{1}_R \right)^q \right)^{1/q} \right\|_{L^p(\nu)}^p$

$\leq \left\| \left(\sum_{R\in\mathcal{E}(Q)} \left(\left(r \fint_Q f\,d\mu \right) \mu(R) a_R \mathbb{1}_R \right)^q \right)^{1/q} \right\|_{L^p(\nu)}^p$ by (2.1)

$= r^p \left(\fint_Q f\,d\mu \right)^p \left\| \left(\sum_{R\in\mathcal{E}(Q)} \left(\left(\int_R \mathbb{1}_Q \, d\mu \right) a_R \mathbb{1}_R \right)^q \right)^{1/q} \right\|_{L^p(\nu)}^p$

$\leq r^p B^p \left(\fint_Q f\,d\mu \right)^p \mu(Q)$ by (1.2) .

Therefore, from inequalities (2.4) and (2.5) we obtain

$$\left\| \left(\sum_{Q\in\mathcal{G}} F_Q^q \right)^{1/q} \right\|_{L^p(\nu)} \leq \left(\sum_{Q\in\mathcal{G}} \|F_Q\|_{L^p(\nu)}^p \right)^{1/p}$$

$$\leq rB \left(\sum_{Q\in\mathcal{G}} \left(\fint_Q d\mu \right)^p \mu(Q) \right)^{1/p}.$$

Apply Theorem 2, taking

$$w_Q = \begin{cases} \mu(Q) & : Q \in \mathcal{G} \\ 0 & : Q \notin \mathcal{G} \end{cases}$$

Equation (2.3) shows that the sequence w_Q satisfies the Carleson measure condition. Hence

$$\sum_{Q \in \mathcal{G}} \left(\fint_Q f d\mu \right)^p \mu(Q) \leq \frac{r}{r-1} (p')^p \|f\|_{L^p(\mu)}^p.$$

Consequently,

$$\left\| M_a^{q,N} f \mu \right\|_{L^p(\nu)}^p \leq \frac{r^{p+1}}{r-1} (p')^p B^p \|f\|_{L^p(\mu)}^p.$$

In particular, since no assumption was made on r other than $r > 1$, one can consider the minimal value of the right hand side constant, which is attained at $\frac{p+1}{p}$. Then

$$\left\| M_a^{q,N} f \mu \right\|_{L^p(\nu)}^p \leq \left(1 + \frac{1}{p} \right)^{p+1} p(p')^p B^p \|f\|_{L^p(\mu)}^p.$$

The right hand side above does not depend on the choice of N. Taking the limit as N approaches ∞ completes the proof for M_a^q. $\qquad\square$

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